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THE EXISTENCE OF RIEMANN INVARIANTS IN THE ONE-DIMENSIONAL EQUATIONS OF THE NON-LINEAR THEORY OF ELASTICITY†

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The conditions for the existence of Riemann invariants of a one-dimensional system of equations of the non-linear theory of elasticity are investigated. Haantjes' diagonalization criterion is used to determine the form of the elastic potential for which the system has six Riemann invariants or three Riemann invariants (for waves which propagate in one direction). In particular, it is shown that the Haantjes criterion is satisfied and there are three Riemann invariants in the case of the elastic potential for slightly-non-linear weakly-anisotropic elastic media [1–3]. A procedure for computing Riemann invariants is described. The Riemann invariants are computed approximately for a form of elastic potential which satisfies the Haantjes criterion.

1. We shall seek simple wave solutions of the system of one-dimensional equations of the theory of elasticity

$$\frac{\partial v^i}{\partial t}=\frac{\partial}{\partial x}\frac{\partial \Phi}{\partial u^i},\quad \frac{\partial u^i}{\partial t}=\frac{\partial v^i}{\partial x}\quad \left(v^i=\frac{\partial^i w}{\partial t},\ u^i=\frac{\partial^i w}{\partial x}\right),\quad i=1,2,3 \tag{1.1}$$

where w^i are the components of the displacement vectors and $\Phi=\Phi(u^1,u^2,u^3)$ is the elastic potential. For simple waves, the components u^i and υ^i satisfy the equations

$$\frac{\partial u^i}{\partial t}+C\frac{\partial u^i}{\partial x}=0,\quad \frac{\partial v^i}{\partial t}+C\frac{\partial v^i}{\partial x}=0 \tag{1.2}$$

where $C=C(u^1,u^2,u^3)$ is the characteristic velocity. It follows from (1.1) and (1.2) that

$$(C^2\delta_{ij}-f_{ij})\frac{\partial u^j}{\partial x}=0,\quad f_{ij}=\frac{\partial^2\Phi}{\partial u^i\partial u^j} \tag{1.3}$$

Equation (1.3) has a non-trivial solution if $\|C^2\delta_{ij}-f_{ij}\|=0$. Hence $\lambda=C^2$ are the eigenvalues of the matrix $F=\|f_{ij}\|$; $\partial u^i/\partial x$ are the components of the eigenvectors of this matrix.

2. We will find a form of the elastic potential $\Phi=\Phi(u^1,u^2,u^3)$ for which system (1.1) has six Riemann invariants. This means that the matrix

$$\| f_{ij}^* \|=\begin{Vmatrix} O & F \\ I & O\end{Vmatrix}$$

(where O is the zero and I the identity 3×3-matrix) can be diagonalized and has only real eigenvalues.

We will use Haantjes' diagonalization criterion [4, 5] in the following form [6]. Let $\upsilon_i^j(u)$ be the components of the tensor (1, 1) on the manifold M with local coordinates $(u^1,\ldots,u^n)$. A Haantjes tensor is a trivalent tensor with components of the form

$$H_{jk}^i=v_s^i v_r^s N_{jk}^r - v_s^i N_{rk}^s v_j^r - v_s^i N_{jr}^s v_k^r + N_{sr}^i v_j^s v_k^r$$

$$N_{ij}^k=v_i^s\frac{\partial v_j^k}{\partial u^s}-v_j^s\frac{\partial v_i^k}{\partial u^s}+v_s^k\frac{\partial v_i^s}{\partial u^j}-v_s^k\frac{\partial v_j^s}{\partial u^i}$$

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where N^k_{ij} are the components of a Nijenhuis tensor. The tensor $\upsilon^j_i(u)$, all of whose eigenvalues are real and different, can be diagonalized in the neighbourhood of a point on the manifold if, and only if, its Haantjes tensor is zero.

For the system of equations (1.2) $\upsilon^j_i(u)$ are the components of the matrix $\| f^*_{ij} \|$, the manifold M is the space region with coordinates u^i, υ^i $(i = 1, 2, 3)$. For any function $\Phi = \Phi(u^1, u^2, u^3)$, since the Nijenhuis tensor is antisymmetric the Haantjes conditions $H^i_{jk} = 0$ comprise 90 third-order equations and are complicated in form. They can be solved by symbolic calculation programs. We have not succeeded in finding a solution of general form and so we will consider only a narrow class of functions Φ.

To simplify the formulae we shall write the superscripts in u^i, $\upsilon^i (i = 1, 2, 3)$ as subscripts, since there are no tensor transformations below.

Suppose the function Φ has the axisymmetric form $\Phi = G(u_1, u_2^2 + u_3^2)$. Consider the component $H^6{}_{22}$ of the Haantjes tensor for the matrix $\| f^*_{ij} \|$

$$H^6_{22} = 16u_3\left((u_2^2+u_3^2)\left(\frac{\partial^2 G}{\partial^2(u_2^2+u_3^2)}\right)^2+\left(\frac{\partial^2 G}{\partial u_1\partial(u_2^2+u_3^2)}\right)^2\right)$$

It follows from the condition $H^6_{22} = 0$ that

$$\Phi = G(u_1, u_2^2+u_3^2) = P(u_1) + b_0(u_2^2+u_3^2), \quad b_0 = \text{const} \tag{2.1}$$

where $P(u_1)$ is an arbitrary function. If the function Φ has the form (2.1), all the equations $H^k_{ij} = 0$ are satisfied. The eigenvalues of the matrix $\| f^*_{ij} \|$ are obviously real. Hence, if the function Φ has the form (2.1), system (1.1) has six Riemann invariants.

We now consider the case where Φ is the sum of arbitrary functions $P(u_1)$ and $f(u_2, u_3)$. Then the equations $H^k_{ij} = 0$ $(i, j, k = 1 - 6)$ reduce to a system of two independent equations for the function

$$H^6_{22} = f_{,33}f_{,223} - f_{,23}f_{,233} - f_{,22}f_{,223} + f_{,222}f_{,23} = 0 \tag{2.2}$$

$$H^6_{23} = f_{,22}f_{,233} - f_{,23}f_{,223} - f_{,33}f_{,233} + f_{,333}f_{,23} = 0 \tag{2.3}$$

$$(f_{,ij} = \partial^2 f / \partial u_i \partial u_j, \quad f_{,ijk} = \partial^3 f / \partial u_i \partial u_j \partial u_k)$$

If $f_{,23} = 0$, then all the equations $H^k_{ij} = 0$ are satisfied and the eigenvalues of the matrix $\| f^*_{ij} \|$ are real and different. We will consider the case $f_{,23} \neq 0$. Then, by adding Eq (2.2) and (2.3), we obtain a partial differential equation for f

$$(h_1(u_3) + h_2(u_2))f_{,23} = 0$$

where h_1 and h_2 are arbitrary functions. Hence, $h_1(u_3) = -h_2(u_2) = C_0 = \text{const}$. The solution of system (2.2), (2.3) is the sum of arbitrary functions f_1 and f_2

$$f(u_2, u_3) = f_1(u_3 - au_2) + f_2(u_2 + au_3), \quad a - 1/a = -C_0$$

Thus, for system (1.1) to have six Riemann invariants, the function Φ must have the form

$$\Phi = P(u_1) + f_1(u_3 - au_2) + f_2(u_2 + au_3)$$

If we make a linear transformation of the variables u_2 and u_3, in the new variables Φ is the sum of three arbitrary functions

$$\Phi = P(u_1) + f_1(u_2) + f_2(u_3) \tag{2.4}$$

Hence, assuming that a function of one variable in Φ can be separated, the equations $H^k_{ij} = 0$ imply that system (1.1) has six Riemann invariants if the other two variables in Φ can also be separated (possibly after a linear transformation of variables), that is, Φ has the form (2.4). It can be shown in this case that all the eigenvalues of the matrix $\| f^*_{ij} \|$ are real and different.

3. We will now find a potential Φ for which, for simple waves, system (1.1) has three Riemann invariants. In that case the matrix of that system F (cf. (1.3)) is symmetric and so has three real eigenvalues $\lambda_{(1)}$, $\lambda_{(2)}$, $\lambda_{(3)}$. Assuming them to be different, we apply the Haantjes criterion. The only non-zero components of the Haantjes tensor H_{ij}^k $(i, j, k = 1, 2, 3)$ and $H_{12}^3, H_{13}^2, H_{21}^3, H_{23}^1, H_{31}^2, H_{32}^1$. The condition that the variable components are equal to zero reduces to one equation for the function Φ. If Φ has the axisymmetric form: $\Phi = G(u_1, u_2^2 + u_3^2)$, this condition is always satisfied and system (1.3) has three Riemann invariants.

We will consider the case when there is weak anisotropy in the u_2, u_3 plane, that is, the function Φ has the form

$$\Phi = G(u_1, u_2^2 + u_3^2) + g_0(u_2^2 - u_3^2), \quad g_0 = \text{const}$$

where g_0 is the anisotropy parameter. We write the equation $H_{12}^3 = 0$ (the fact that the other non-zero component of the Haantjes tensor are zero leads to the same equation)

$$H_{12}^3 = A(u_1, u_2, u_3)g_0 + B(u_1, u_2, u_3)g_0^2 + C(u_1, u_2, u_3)g_0^3 = 0$$

The functions A, B and C are independent of g_0, and it follows from the last equation that if

$$A(u_1, u_2, u_3) = 0, \quad B(u_1, u_2, u_3) = 0, \quad C(u_1, u_2, u_3) = 0 \tag{3.1}$$

the Haantjes criterion is satisfied. The expression for the functions A, B and C have the form

$$\begin{aligned} &A(u_1, u_2, u_3) = \xi_1\varphi(u_1, u_2, u_3) \\ &B(u_1, u_2, u_3) = -\xi_1\eta \\ &C(u_1, u_2, u_3) = -64u_2u_3\xi_2 \end{aligned} \tag{3.2}$$

$$\xi_1 = \frac{\partial^2 G}{\partial u_1 \partial(u_2^2 + u_3^2)}, \quad \eta = \frac{\partial^3 G}{\partial^3(u_2^2 + u_3^2)}, \quad \xi_2 = \frac{\partial^3 G}{\partial u_1 \partial^2(u_2^2 + u_3^2)}$$

The form of the function $\varphi(u_1, u_2, u_3)$ is too complicated to give here.

It follows from expressions (3.2) that if $\xi_1 = 0$, then Eqs (3.1) are valid and the Haantjes diagonalization condition is satisfied. Hence, the function Φ for the elastic potential has the form

$$\Phi = P(u_1) + Q(u_2^2 + u_3^2) + g_0(u_2^2 - u_3^2)$$

where P and Q are arbitrary functions.

If $\xi_2 = 0$, $\eta = 0$, we have

$$A(u_1, u_2, u_3) \neq 0, \quad B(u_1, u_2, u_3) = 0, \quad C(u_1, u_2, u_3) = 0$$

and the elastic potential function can be written in the form

$$\Phi = A_0(u_2^2 + u_3^2)^2 + N(u_1)(u_2^2 + u_3^2) + M(u_1) + g_0(u_2^2 - u_3^2) \tag{3.3}$$

where A_0 is an arbitrary constant, $M(u_1)$ and $N(u_1)$ are functions which can be determined by substituting the second derivatives of the function $\Phi = \Phi(u^1, u^2, u^3)$ of the form (3.3) into the equation $H_{12}^3 = 0$. As a result we obtain a polynomial in the variables u_2 and u_3. From the fact that the coefficients of powers of the variables u_2 and u_3 are equal to zero, we obtain two differential equations for the functions $N(u_1)$ and $M(u_1)$

$$N'g_0(-64A_0^2 + 24A_0N'' - 2(N'')^2 + N'N''') = 0 \tag{3.4}$$

$$N'g_0(-16A_0N - 6(N')^2 + 4NN'' + 8A_0M'' - 2N''M'' + N'M''') = 0 \tag{3.5}$$

These equations have two obvious solutions: $g_0 = 0$, that is $\Phi = G(u_1, u_2^2 + u_3^2)$, and $N(u_1) = N_1 =$ const, which is the form of the elastic potential function given in [1–3].

If $N(u_1) \neq$ const, we will solve the equation corresponding to the case where the expression in brackets in Eq. (3.4) vanishes. Using the replacement of variables $N'' = f(N')$, $N' = x \neq$ const (if $N' =$ const, then $A_0 = 0$) and introducing the variable $\tau = \ln x$, we write this equation in the form

$$df/d\tau = \chi(f)/f, \quad \chi(f) = 2f^2 - 24A_0f + 64A_0^2 \tag{3.6}$$

The solutions of Eq. (3.6) have the form

$$f_1 = 8A_0,\ f_2 = 4A_0 \quad (\chi(f) = 0)$$

$$f_{3,4} = 8A_0 + \frac{x^2}{2k_0^2} \pm 2\frac{x}{k_0}\sqrt{A_0 + \frac{x^2}{16k_0^2}} \quad (\chi(f) \neq 0)$$

$$k_0 = \text{const}$$

The roots f_1, f_2, f_3 correspond to three series of solutions of Eq. (3.4)

$$N(u_1) = 4A_0 u_1^2 + b_1 u_1 + b_0 \tag{3.7}$$

$$N(u_1) = 2A_0 u_1^2 + b_1 u_1 + b_0 \tag{3.8}$$

$$b_1 = \text{const},\quad b_0 = \text{const}$$

$$N(u_1) = 2A_0(u_1 - k_2)^2 - k_0^2 \ln(u_1 - k_2) + k_3,\quad k_3 = \text{const} \tag{3.9}$$

The constant k_2 is associated only with the shift with respect to u_1 and so hence forth we will assume that $k_2 = 0$.

Substituting expressions (3.7), (3.8) and (3.9) for the function $N(u_1)$ into Eq. (3.5), we obtain three differential equations for the function $M(u_1)$, the solutions of which are the corresponding series of expressions for the functions $M(u_1)$

$$M(u_1) = -\frac{10}{3}A_0 u_1^4 + c_3 u_1^3 + c_2 u_1^2 + c_1 u_1 + c_0 \tag{3.10}$$

$$c_2 = \frac{1}{8A_0}(-8A_0 b_0 + 3b_1 c_3 + 3b_1^2)$$

$$c_3 = \text{const},\quad c_1 = \text{const},\quad c_0 = \text{const}$$

$$M(u_1) = A_0 u_1^4 + \frac{b_1}{6}u_1^3 + c_2 u_1^2 + c_1 u_1 + c_0$$

$$c_2 = \text{const},\quad c_1 = \text{const},\ c_0 = \text{const} \tag{3.11}$$

$$M(u_1) = A_0 u_1^4 - 2A_0 u_1^2 c_1 + u_1 c_2 - k_0^2 u_1^2 \ln u_1 - \frac{1}{4A_0}(-k_0^4 - 4A_0 k_0^2 c_1)\ln u_1 + c_3 \tag{3.12}$$

$$c_1 = \text{const},\ c_2 = \text{const},\ c_3 = \text{const},\ k_0 = \text{const}$$

The functions $N(u_1)$ (3.7), (3.8), (3.9) and $M(u_1)$ of (3.10), (3.11), (3.12) and expression (3.3) determine the specific form of the elastic potential Φ for which system (1.3) has three Riemann invariants.

4. We will now consider the special case when the elastic potential function Φ has the form

$$\Phi = R^4 + g_0(u_2^2 - u_3^2),\quad R^2 = u_1^2 + u_2^2 + u_3^2 \tag{4.1}$$

(we have put $A_0 = 1$ in (3.3), $b_0 = 0, b_1 = 0$ in (3.8), and $c_0 = 0, c_1 = 0, c_2 = 0$ in (3.11)).

In this case if $g_0 = 0$ ($\Phi = R^4$) we define the eigenvalues of the matrix F as follows: $\lambda_{(1)} = 12R^2$, $\lambda_{(2,3)} = 4R^2$. The eigenvectors $\mathbf{a}_{(1)}, \mathbf{a}_{(2)}, \mathbf{a}_{(3)}$ with components

$$\mathbf{a}_{(1)} = (u_1, u_2, u_3),\ \mathbf{a}_{(2)} = (-u_3, 0, u_1),\ \mathbf{a}_{(3)} = (-u_2, u_1, 0)$$

correspond to these eigenvalues, and the Riemann invariants of system (1.1) have the form $I_{(1)} = H_1(R^2)$, $I_{(2)} = H_2(u_1/u_3)$, $I_{(3)} = H_3(u_1/u_2)$, where H_1, H_2, H_3 are arbitrary functions.

If $g_0 \neq 0$, the eigenvalues $\lambda_{(1)}, \lambda_{(2)}, \lambda_{(3)}$ of the matrix F are found from the equation

$$-\sigma^3 + 8\sigma^2(u_1^2 + u_2^2 + u_3^2) - 32u_1^2 g_0^2 + 4\sigma g_0(4u_2^2 - 4u_3^2 + g_0) = 0$$

$$\lambda = \sigma + 4(u_1^2 + u_2^2 + u_3^2)$$

To each eigenvalue $\lambda_{(i)}$ there corresponds an eigenvector whose components x_1, x_2, x_3 are given by the formulae

$$\begin{aligned} &x_1 = u_1/u_2 - 2u_1g_0/(\sigma u_2), \; x_2 = 1 \\ &x_3 = \sigma/(8u_2u_3) - u_1^2/(u_2u_3) - u_2/u_3 - g_0/(4u_2u_3) + 2u_1^2 g_0/(\sigma u_2u_3) \end{aligned} \tag{4.2}$$

If g_0 is small but non-zero, the eigenvalues have the form

$$\begin{aligned} &\lambda_{(1)} = 12R^2 + 2(u_2^2 - u_3^2)g_0/R^2 \\ &\lambda_{(2,3)} = 4R^2 + Dg_0 \\ &D = \left[(u_3^2 - u_2^2) \pm \sqrt{(u_3^2 - u_2^2)^2 + 4u_1^2R^2}\right]/R^2 \end{aligned} \tag{4.3}$$

We will find the Riemann invariants of system (1.1) in the case when Φ has the form (4.3) and g_0 is small. The hyperbolic system (1.1), allowing for Eqs (1.2), can be written in the form

$$C_{(k)}\frac{\partial u_i}{\partial t} + f_{ij}\frac{\partial u_j}{\partial x} = 0, \quad f_{ij} = \frac{\partial \Phi}{\partial u_i \partial u_j}, \quad k = 1,2,3 \tag{4.4}$$

where $C_{(k)} = C_{(k)}(u_1, u_2, u_3)$ is the characteristic velocity and the subscript in brackets denotes the characteristic velocity corresponding to the eigenvalue with the same subscript. Henceforth, any value for which there is a subscript in brackets corresponds to the eigenvalue with the same subscript.

We multiply the equations of system (4.4) on the left by the eigenvector $\mathbf{a}_{(k)} = (a_{1(k)}, a_{2(k)}, a_{3(k)})$ of the matrix F (the subscript without brackets denotes the number of the eigenvector component). We obtain

$$a_{i(k)}C_{(k)}\frac{\partial u_i}{\partial t} + \lambda_{(k)}a_{j(k)}\frac{\partial u_j}{\partial x} = 0$$

where summation is carried out over repeated subscripts without brackets and $\lambda_{(k)} = (C_{(k)})^2$ is an eigenvalue of the matrix F. If an integration coefficient $\mu_{(k)}$ is found for each eigenvalue such that

$$\mu_{(k)}a_{i(k)} = \frac{\partial I_{(k)}}{\partial u_i} \tag{4.5}$$

then system (4.4) can be written in the form

$$\frac{\partial I_{(k)}}{\partial t} + C_{(k)}\frac{\partial I_{(k)}}{\partial x} = 0$$

Eliminating $\mu_{(k)}$ from Eqs (4.5) we obtain a system of two partial differential equations for determining the Riemann invariants $I_{(k)}$

$$\frac{\partial I_{(k)}}{\partial u_1} = \frac{a_{1(k)}}{a_{2(k)}}\frac{\partial I_{(k)}}{\partial u_2}, \quad \frac{\partial I_{(k)}}{\partial u_3} = \frac{a_{3(k)}}{a_{2(k)}}\frac{\partial I_{(k)}}{\partial u_2}, \quad k = 1,2,3 \tag{4.6}$$

System (4.6) for the first Riemann invariant $I_{(1)}$, taking into account expressions (4.2) and (4.3) for the components of the eigenvector $\mathbf{a}_{(1)} = (a_{1(1)}, a_{2(1)}, a_{3(1)})$, can be written as follows:

$$\frac{\partial I_{(1)}}{\partial u_1^2} + \left(\frac{1}{4R^2}g_0 - 1\right)\frac{\partial I_{(1)}}{\partial u_2} = 0, \quad \frac{\partial I_{(1)}}{\partial u_3^2} + \left(\frac{1}{2R^2}g_0 - 1\right)\frac{\partial I_{(1)}}{\partial u_2} = 0 \tag{4.7}$$

We will seek a solution of system (4.7) in the approximate form

$$I_{(1)} = H_1(R) + g_0T_{(1)} + O(g_0^2) \tag{4.8}$$

where $H_1(R)$ is the first Riemann invariant when $g_0 = 0$. From (4.7) we obtain a system of equations for $T_{(1)}$, the solution of which is

$$T_{(1)} = \Phi_1(R) + (u_2^2 - u_3^2)\frac{1}{8R^3}\frac{\partial H_1(R)}{\partial R} \tag{4.9}$$

where $\Phi_{(1)}$ is an arbitrary function.

Hence, we have found the first Riemann invariant (expressions (4.8) and (4.9)) of system (1.1) in the case where Φ has the form (4.1) and g_0 is small but non-zero.

Because the expressions for the components of eigenvectors $\mathbf{a}_{(2)}$ and $\mathbf{a}_{(3)}$ (4.2) are so complicated, $I_{(2)}$ and $I_{(3)}$ were found by representing the components of vectors $\mathbf{a}_{(2)}$ and $\mathbf{a}_{(3)}$ in the form of series in u_1. System (4.6) for $I_{(2)}$ then has the form

$$\frac{\partial I_{(2)}}{\partial u_1}=-\frac{2u_1u_2}{u_3^2-u_2^2}\frac{\partial I_{(2)}}{\partial u_2},\quad \frac{\partial I_{(2)}}{\partial u_3}=\left(-\frac{u_2}{u_3}+\frac{2u_2u_1^2}{u_3(u_3^2-u_2^2)}\right)\frac{\partial I_{(2)}}{\partial u_2} \tag{4.10}$$

We will seek a solution of system (4.10) in the form

$$I_{(2)}=T_{(2)}+u_1^2S_{(2)},\quad T_{(2)}=T_{(2)}(u_2,u_3),\quad S_{(2)}=S_{(2)}(u_2,u_3) \tag{4.11}$$

From the second equation of (4.10) we then obtain

$$T_{(2)}=\Phi_2(u_2/u_3) \tag{4.12}$$

where $\Phi_{(2)}$ is an arbitrary function.

The first equation of (4.10) gives

$$S_{(2)}=-u_2\Phi_2'(u_2/u_3)/[u_3(u_3^2-u_2^2)] \tag{4.13}$$

Thus, expressions (4.1)–(4.13) give the value of the second Riemann invariant, apart from linear terms in g_0 and quadratic terms in u_1.

For the third Riemann invariant, taking into account the expressions for the components of the eigenvector $\mathbf{a}_{(3)}$ (4.2) for small u_1, we can write (4.6) as follows:

$$\begin{aligned}&\frac{\partial I_{(3)}}{\partial u_1}=\left(\frac{u_3^2-u_2^2}{u_1u_2}+\frac{2u_3^2u_1}{u_2(u_3^2-u_2^2)}\right)\frac{\partial I_{(3)}}{\partial u_2}\\&\frac{\partial I_{(3)}}{\partial u_3}=\left(-\frac{u_2}{u_3}-\frac{2u_3u_1^2}{u_2(u_3^2-u_2^2)}\right)\frac{\partial I_{(3)}}{\partial u_2}\end{aligned} \tag{4.14}$$

We will seek a solution of system (4.14) in the form

$$I_{(3)}=T_{(3)}(1+u_1^2S_{(3)}+\ldots),\quad T_{(3)}=T_{(3)}(u_1,u_2,u_3),\quad S_{(3)}=S_{(3)}(u_2,u_3) \tag{4.15}$$

Then from (4.14) in the highest approximation we obtain a system of equations for $T_{(3)}$, which we solve to give

$$T_{(3)}=u_1^{2m}(u_3^2-u_2^2)^{-m},\quad m-\text{is an integer} \tag{4.16}$$

Substituting expressions (4.15) and (4.16) into the first equation of (4.14), we find the correction $S_{(3)}$

$$S_{(3)}=-2mu_3^2(u_3^2-u_2^2)^{-2}+C_1(u_3^2-u_2^2)^{-1} \tag{4.17}$$

where C_1 is an arbitrary constant. Expressions (4.15)–(4.17) give an approximate value for the third Riemann invariant $I_{(3)}$.

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